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ON THE IDENTICAL RELATIONS BETWEEN THE DETERMINANTS OF AN ARRAY.

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The object of this paper is a systematic account of the subject designated by Muir “vanishing aggregates of determinant products.[*] It will be seen that they are in general Laplacian expansions of vanishing determinants, which do not vanish when regarded as functions of the minor determinants. They include as a special case the fundamental determinant identity, the vanishing determinant with two identical rows or columns. This case (historically Vandermonde's identity[†]) underlies all the others. The first example of a more complicated kind is due to Bezout[‡], who gave among the determinants of the 6×3 array the relation: $(123)(456)-(124)(356)+(134)(256)-(234)(156)=0$. Desnanot[§] gave a number of new forms and the general method of deriving new from the old, called now the method of the extensional. More recently Cayley[||], Sylvester,[¶] and Muir[°] have considered the general forms, while Kronecker,['] Runge[″], and others have studied the special forms relating to an axisymmetric determinant.

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[*] *The Theory of Determinants*, p. 52.

[†] Ib., p. 22.

[‡] Ib., p. 52.

[§] Muir, p. 143.

[||] *Coll. Math. Papers*, 1. 55.

[¶] *Phil. Mag.*, 1850.

[°] *Proc. R. S. E.*, 1891, p. 73.

['] Sitzungsb, *d. k. Akad. d. Wiss.*, 1882.

[″] Crelle, 93—319.

We consider the rectangular array whose elements are

$$a_{ik} \quad (i=1,\ 2,\ \ldots,\ m;\ k=1,\ 2,\ \ldots,\ n;\ m>n).$$

The transformation

$$a'_{ij}=\Sigma_k\, \lambda_{jk}\, a_{ik} \qquad |\, \lambda_{jk}\, | =1 \quad (j=1,\ \ldots,\ n)$$

leaves all determinants of n rows unaltered, while introducing n^2-1 arbitrary quantities among the elements. There are mn elements and $\binom{m}{n}$ determinants, hence at least $\binom{m}{n}-(mn-n^2+1)$ relations exist between the determinants which are identities in the elements. No more independent relations exist; for we can assign to $mn-n^2+1$ determinants arbitrary values as follows: The determinant formed by prefixing one column of the set to the array of any $n+1$ rows vanishes identically. When expanded by elements of the prefixed columns and their minors. This gives the fundamental determinant identities

$$\sum_{i=1}^{n+1} a_{ik}\, (a_{11}\, a_{22} \cdots\cdots a_{i-1.\, i-1}\, a_{i+1,\, i} \cdots\cdots a_{n+1,\, n}) \equiv 0,$$

one for each $k=1,\ 2,\ \ldots,\ n$. There are in all $n\binom{m}{n+1}$ such identities. They are lineo-linear in the elements and determinants of order n, homogeneous in elements, determinants, rows, and columns, and contain $n+1$ terms. We shall call them the *lineo-linear relations.*

Postulating that not all the determinants vanish, suppose that $(1,\ 2,\ 3,\ \ldots,\ n)\neq 0$. In this determinant give arbitrary values to n elements of $n-1$ columns and $n-1$ elements of the nth column provided that the minor of the remaining element be not zero. This last element can then be chosen uniquely so as to give $(1,\ 2,\ 3,\ \ldots,\ n)$ its assigned value. The lineo-linear relations may now be written $a_{n+j,\, k}\, (1,\ 2,\ \ldots,\ n)=-a_{n,\, k}\, (1,\ 2,\ \ldots,\ n-1,\ n+j)-\ldots =0$, where j specifies any new row. There are $n(m-n)$ such relations and the determinant on the left and the elements on the right are assigned. Giving assigned values to the $n(m-n)$ determinants on the right we have unique values for the remaining $n(m-n)$ elements. They are finite for $(1,\ 2,\ \ldots,\ n)\neq 0$. All the elements and hence all the determinants are now assigned. The number of independent relations being at least, and not more than, is exactly

$$I=\binom{m}{n}-(mn-n^2+1).$$

In general however an indefinite number of relations exists, which may be reduced by definition and classification to a finite number R, all of which must be satisfied. Except for some special cases, $R>I$ and a *plexus* exists. That is, in any special case some I of the R being satisfied the others follow by virtue of relations among the R which are identities in the determinants. We proceed to the classification.

There exists no relation which is not either itself homogeneous in the elements of a certain set of rows, and of a certain set of columns, or obtainable by the composition of relations having these properties. The most general relation can be regarded as linear in the determinants of some set and as having for coefficients rational integral functions of the elements. Let $\Sigma C \triangle \equiv 0$ be such a relation. Choose a determinant containing a set of columns $(i, j, k, \ldots\ldots\ldots)$. Multiply these columns in the array by $s_i, s_j, s_k, \ldots\ldots\ldots$ The result must vanish for every s. Hence equating to zero the coefficients of the various powers and products of the s's we obtain a set of identities

$$\Sigma C_c \triangle_c = 0$$

homogeneous in a certain set of columns. A similar process in the rows gives a set of identities $\Sigma C_{cr} \triangle_{cr} = 0$, when each term contains the same set of rows and columns. Without loss of generality we may set $m \geq n$, where there are m rows and n columns. The original identity depends on and is composed of these. Since a single determinant cannot vanish identically the coefficient of each determinant must contain an element of some row not represented in this determinant but in others.

First consider possible relations between the $n+1$ determinants which enter a single set of $n+1$ rows. These can be obtained from a vanishing determinant whose jth row is

$$A_{j_1} a_{j_1} + A_{j_2} a_{j_2} + \ldots\ldots\ldots A_{j_n} a_{j_n}, \quad a_{j_1}, a_{j_2}, a_{j_3}, \ldots\ldots, a_{j_n}.$$

Consider the special case $a_{1k}=1$, $a_{2k}=a^k$, $a_{3k}=b^k$, $\ldots\ldots$, $a_{n+1,k}=z^k$, and put first $a=1$. We have

$$(A_{11}+A_{12}+\ldots\ldots\ldots)[(1, b, c^2, \ldots\ldots, z^n)]=(A_{21}+A_{22}+\ldots\ldots\ldots)[(1, b, c^2, \ldots\ldots, z^n)]$$

or $\Sigma A_{1k}=\Sigma A_{2k}$, for $(1, b, c^2, \ldots\ldots, z^n) \neq 0$. So in general $\Sigma A_{jk}=\Sigma A_{1k}$.

Returning to the general form, the A_{jk} each lack a kth column of being homogeneous in columns and a jth row of being homogeneous in rows. Hence, if in $\Sigma A_{1k}=\Sigma A_{2k}$ we place all the elements of the 1st column$=0$ we obtain $A_{11} = A_{21}$, and similarly $A_{jk}=A_{1k}$, and the relation reduces to a series of multiples of the lineo-linear relations.

Secondly we treat the case where determinants of more than $n+1$ rows enter.

Lemma. Out of m different things r sets of n things are taken, and for each of these sets of n, every set of $n+1$ including it is formed. If the number of these sets is s and $m-n \geq 2$ and *$m \geq 2$*, then $ns > r$.

Let the r sets be $a_{j_1}, a_{j_2}, a_{j_3}, \ldots\ldots, a_{j_n}$ $(j=1, 2, \ldots\ldots, n)$. Form the sets

$$(a_{j_1}, a_{j_2}, \ldots\ldots, a_{j_n})a_t \quad (j=1, 2, \ldots\ldots, r;\ t=1, 2, \ldots\ldots, n).$$

Of these nr have repeated elements leaving $(m-n)r$ which may in the complete system occur any number of times up to $n+1$. If T_f be the number occurring f times, $s=\Sigma \frac{T_f}{f}$ $(f=1, \ldots, n+1)$; $\Sigma T_f=(m-n)r$. Therefore

$$ns=\Sigma\frac{nT_f}{f}>\frac{n(m-n)r}{n+1}>r.$$

Now let determinants of more than one set of $n+1$ rows enter the relation. If s be the number of these sets from the ns lineo-linear relations belonging to the sets, multiply them by indeterminate coefficients, add and equate the coefficient of each determinant to the corresponding coefficient in the given relation. There are r linear homogeneous equations between ns indeterminates. They are independent; for, each involves a given set of rows, and as $ns>r$, rational integral functions of the elements can be found (in general in more than one way) for the indeterminates. This reduces the given relation to a sum of multiples of the lineo-linear relations.

We consider primarily then "vanishing aggregates of determinant products" of the second order. Those of the third order vanish only by being sums of products of determinants by such second order aggregates. The third order aggregates of which Monge* gave the first example, are important as relations used in elucidating the plexus between those of those of the second order.

A vanishing aggregate of determinant products of the second order, is then homogeneous in rows and columns, of order $2n$ in the elements. It contains at least three terms. For, the product of the determinants containing the rows $p, q, r, \ldots, w$ and $s, t, u, \ldots, z$, respectively, has a term

$$a_p\, b_q\, c_r, \ldots, l_w \,.\, a_s\, b_t\, c_u, \ldots, l_z$$

which can only be cancelled by a product from determinants such as $(p, t, r, \ldots, w)(s, q, u, \ldots, z)$ formed by simple interchange from first product, and this in turn has a term $a_t\, b_p\, c_r, \ldots, l_w \,.\, a_q\, b_s\, c_u, \ldots, l_z$ which does not occur in the first product.

We take a set of R such relations $(R \geq I)$ given by systematic enumeration and such that in every case these R are necessary and sufficient relations, but in any particular case (*i. e.* some one determinant not zero) a definite I of these relations are the independent ones. The set R may then be considered as fundamental.

First we have relations of $n+1$ terms given by the expansion of the identically vanishing determinant

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*Muir, p. 68.

$$\begin{vmatrix} a_1 & b_1 & c_1 \ldots\ldots l_1, & 0 & 0 & 0 \ldots\ldots 0 \\ a_2 & b_2 & c_2 \ldots\ldots l_2, & 0 & 0 & 0 \ldots\ldots 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1} & b_{n-1} & c_{n-1} \ldots\ldots l_{n-1}, & 0 & 0 & 0 \ldots\ldots 0 \\ a_n & b_n & c_n \ldots\ldots l_n, & a_n & b_n & c_n \ldots\ldots l_n \\ a_{n+1} & b_{n+1} & c_{n+1} \ldots\ldots l_{n+1}, & a_{n+1} & b_{n+1} & c_{n+1} \ldots\ldots l_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{2n} & b_{2n} & c_{2n} \ldots\ldots l_{2n}, & a_{2n} & b_{2n} & c_{2n} \ldots\ldots l_{2n} \end{vmatrix}$$

As the choice of rows and of rows set opposite zeros is alone material we may denote this by

$$\begin{array}{cccccc|cccc} 0, & 0, & 0, & \ldots, & 0, & n & n+1, & n+2, & \ldots, & 2n \\ \hline 1, & 2, & 3, & \ldots, & n-1, & n & n+1, & n+2, & \ldots, & 2n \end{array}$$

There are $\binom{m}{2n}\binom{2n}{n-1}$ of these, except for $n=2$ where the relation remains unchanged whichever row is placed opposite the zero. For $n=2$ the number is $\binom{m}{4}$.

This class of relations differs from the other classes to be discussed in important points. Since all the rows are different each determinant is multiplied by the same complementary determinant in each of the $\binom{2n}{n-1}$ relations formed from a given $2n$ rows. We may refer to this class as the *complementary class.*

Next we have relations of r terms $2<r<n+1$, given by the expansion of a similar vanishing determinant whose symbol is

$$\begin{array}{ccccc|ccccccccc} 0, & 0, & \ldots, & 0, & n & n+1, & n+2, & \ldots, & n+r-1, & 1, & 2, & \ldots, & n-r+1 \\ \hline 1, & 2, & \ldots, & n-1, & n & n+1, & n+2, & \ldots, & n+r-1, & 1, & 2, & \ldots, & n-r+1. \end{array}$$

The number is

$$\binom{n}{n+r-1}\binom{n+r-1}{n-r+1}\binom{2r-2}{r-2},$$

except for $r=3$ when the last factor is replaced by 1. The total for $r=3, 4, \ldots, n+1$ is

$$R=\binom{m}{n-1}\left[\binom{m}{n+1}-\frac{(m-n+1)(m-n)(mn-n^2-m+5)}{8}\right].$$

For $r<n+1$ the relations are "extensionals" of the relations of the complementary class for $m'=2n'$, where m' is the number of rows not used as "extenders." For example, $\frac{0\ 0\ 0\ 4\ |\ 5\ 6\ 7\ 1}{1\ 2\ 3\ 4\ |\ 5\ 6\ 7\ 1}$ may be regarded as derived from $\frac{0\ 0\ 4\ |\ 5\ 6\ 7}{2\ 3\ 4\ |\ 5\ 6\ 7}$ by using 1 as an "extender."

If there is an identical relation between complementary relations involving the same set of rows of the form $R_1+R_2+R_3=\dots\dots=0$, this relation may be extended by introducing any set of new rows into all the determinants entering. For example, the relation of complementary type of the 3×6 array are connected (among other connections) by

$$\frac{0\ 0\ 1 \mid 4\ 5\ 6}{1\ 2\ 3 \mid 4\ 5\ 6}+\frac{0\ 0\ 1 \mid 2\ 5\ 6}{3\ 4\ 1 \mid 2\ 5\ 6}+\frac{0\ 0\ 1 \mid 2\ 3\ 4}{5\ 6\ 1 \mid 2\ 3\ 4}\equiv 0,$$

which vanishes in the determinants. The relation

$$\frac{0\ 0\ 0 \dots\dots 0\ 0\ 3 \mid 4\ 5\ 6\ a\ b\ c \dots\dots}{a\ b\ c \dots\dots 1\ 2\ 3 \mid 4\ 5\ 6\ a\ b\ c \dots\dots}+\frac{0\ 0\ 0 \dots\dots 0\ 0\ 1 \mid 2\ 5\ 6\ a\ b\ c \dots\dots}{a\ b\ c \dots\dots 3\ 4\ 1 \mid 2\ 5\ 6\ a\ b\ c \dots\dots}$$

$$+\frac{0\ 0\ 0 \dots\dots 0\ 0\ 1 \mid 2\ 3\ 4\ a\ b\ c \dots\dots}{a\ b\ c \dots\dots 5\ 6\ 1 \mid 2\ 3\ 4\ a\ b\ c \dots\dots}\equiv 0$$

also holds, when $a\ b\ c \dots\dots$ is any set of rows. For, the vanishing of the simple relation depends on interchanges of rows, which interchanges are not affected by the extension.

Given any relation of complementary type

$$\frac{0\ 0\ 0 \dots\dots (r-1) \mid r,\ r+1,\ \dots,\ 2(r-1)}{1\ 2\ 3 \dots\dots (r-1) \mid r,\ r+1,\ \dots,\ 2(r-1)},$$

involving r terms, we obtain by extension $\binom{m-2r+2}{n-r+1}$ new relations. There are $\binom{m}{2(r-1)}\binom{2(r-1)}{r-2}$ such complementary relations, so that

$$\binom{n-r+1}{m-2r+2}\binom{m}{2(r-1)}\binom{2(r-1)}{r-2}=\binom{m}{n+r-1}\binom{n+r-1}{n-r+1}\binom{2r-2}{r-2}$$

is the number of relations of r terms in general.

The number of relations involving a given determinant is

$$K=\sum_{r=4}^{r=n+1}\left[2(r-1)\binom{m-n}{r-1}\binom{n}{r-1}\right]+\binom{m-n}{2}\binom{n}{2}.$$

For three-termed relations, we have the $(n-2)$nd extensions of the single relation $\frac{0\ b \mid c\ d}{a\ b \mid c\ d}$. $\binom{m-n}{2}$ is a factor of the number for the two rows in the given determinant, and $\binom{n}{2}$ a factor for choice of two rows not in the extenders. For $r>3$ we have a factor for the order of arrangement of $\frac{0\ 0 \dots\dots c \mid d\ e \dots\dots f}{a\ b \dots\dots c \mid d\ e \dots\dots f}$. If

a, b, ..., c belong to the given determinant, $(r-2)$ or none of these rows occur opposite the zeros. This gives $2(r-1)$ as the factor, and K as the total.

The relations of more than three terms involving a given determinant k are not independent but are connected with one another and those of their own type not involving k by linear relations whose coefficients are ± 1.

Take first the case $m=2n$ and the relations of $n+1$ terms. Write the $2n$ column numbers as

$$1,\ 2,\ 3,\ \ldots\ldots,\ n,\ \overline{n},\ (\overline{n-1}),\ \ldots\ldots,\ \overline{3},\ \overline{2},\ \overline{1}.$$

The relation

$$\begin{array}{l|l} 0,\ 0,\ 0,\ \ldots\ldots\ldots\ldots,\ 0,\ n & \overline{n},\ (\overline{n-1}),\ \ldots\ldots,\ \overline{2},\ \overline{1} \\ 1,\ 2,\ 3,\ \ldots\ldots,\ (n-1),\ n & \overline{n},\ (\overline{n-1}),\ \ldots\ldots,\ \overline{2},\ \overline{1} \end{array}$$

contains $k=(1,\ 2,\ \ldots\ldots,\ n)$. So does each relation arising out of this by any interchanges of $1,\ 2,\ \ldots\ldots,\ n$; if $\overline{n},\ (\overline{n-1}),\ \ldots\ldots,\ \overline{2},\ \overline{1}$ are interchanged also, r and $\overline{r}$ going together, the leading terms have all the same sign. In general these interchanges do not alter the signs. The corresponding relation

$$\begin{array}{l|l} 0,\quad 0,\ \ldots\ldots\ldots\ldots,\ 0,\ 1 & 2,\ 3,\ \ldots\ldots,\ n,\ \overline{n} \\ (\overline{n-1}),\ (\overline{n-2}),\ \ldots\ldots,\ \overline{1},\ 1 & 2,\ 3,\ \ldots\ldots,\ n.\ \overline{n} \end{array}$$

and all the results of interchanges also contain k. If $n>2$ none of this set falls in the last mentioned. Add all these relations; the result is not an identity in the relations but vanishes in the determinants. For example, the products $[1,\ 2,\ \ldots\ldots,\ (n-1),\ \overline{a}][n,\ \overline{n},\ \ldots\ldots,\ (\overline{a+1}),\ (\overline{a-1}),\ \ldots\ldots,\ \overline{1}]$ and $[(\overline{n-1}),\ \ldots\ldots,\ (\overline{a+1}),\ \overline{a},\ (\overline{a-1}),\ \ldots\ldots,\ \overline{1},\ 1][2,\ 3,\ \ldots\ldots,\ n,\ \overline{n}]$ occur in the two sets. By the interchange $(1n)$ followed by $(1a)$, the latter becomes

$$[(\overline{n-1}),\ \ldots\ldots,\ (\overline{a+1}),\ \overline{1},\ (\overline{a-1}),\ \ldots\ldots,\ \overline{2},\ \overline{n}]$$
$$[2,\ 3,\ \ldots\ldots,\ (a-1),\ 1,\ (a+1),\ \ldots\ldots,\ (n-1),\ a,\ \overline{a}]$$

which is identical, except as to sign, with the first product. By suitable interchanges this pair of terms becomes other pairs and exhausts the two sets together. Hence we have an identity which may be written $\Sigma R(k)=0$ $(m=2n)$. A single interchange of a barred and unbarred column changes all the $R(k)$'s but two to $R(\overline{k})$'s, the exception being the two free from the interchanged columns. This interchange can be chosen so that for any one of the first set we have any one of the second left unchanged, and by subtraction we have for every pair of $R(k)$'s:

$$R_1(k)+R_2(k)+\Sigma R(\overline{k})=0\ (m=2n).$$

This relation can now be extended by any set of columns. As no such relations hold for the 2×4 array the extended relations hold only down to R_4.

The reduction of the $R(\overline{k})$ to dependence on the $R(k)$ can be accomplished as follows. Let

$$R(\bar{k})=\frac{0\ 0\ 0\ 0\ldots\ldots z \ \big|\ \alpha\ \beta\ \gamma\ldots\ldots\varsigma}{a\ b\ c\ d\ldots\ldots z\ \big|\ \alpha\ \beta\ \gamma\ldots\ldots\varsigma},\ k=(1,\ 2,\ \ldots,\ n).$$

Form the $3n$ row determinant whose upper right $2n\times 2n$ places are occupied by $R(\bar{k})$ and lower left $n\times n$ by k and fill in zeros below the zeros of $R(\bar{k})$ letters below the letters. Repeat k twice above itself. The symbol of it may be taken as

$$\begin{array}{l|l|l}
1\ 2\ldots\ldots n & 0\ 0\ 0\ldots\ldots 0\ z & \alpha\ \beta\ \gamma\ \delta\ldots\ldots\varsigma \\
1\ 2\ldots\ldots n & a\ b\ c\ldots\ldots y\ z & \alpha\ \beta\ \gamma\ \delta\ldots\ldots\varsigma \\
1\ 2\ldots\ldots n & 0\ 0\ 0\ldots\ldots 0\ z & \alpha\ \beta\ \gamma\ \delta\ldots\ldots\varsigma
\end{array}$$

Expanding by lower n rows, we get $D=kR(\bar{k})+\Sigma\, a\, R$, when every R has at least one column which occurs in k. This determinant vanishes identically in the elements but not necessarily in the determinants. Expanding by the middle n rows, we see that only the determinants $[a\, b\, c\ldots\ldots y\, X]$ have non-vanishing coefficients. If X comes from the columns to the right the coefficient is 0 for n odd, and $2\,R(k)$ for n even. If X comes from k the coefficient contains $(n-1)$ columns of k if it does not vanish, and is expressible as $\Sigma R'_{n-1}$. We may then write

$$k\,R(\bar{k})\equiv\Sigma\, a\, R_1+\Sigma\, b\, R_{n-1}+\Sigma\, c\, R(k)$$

when every R_1 contains one column of k not opposite a zero. R'_{n-1} has $(n-1)$ columns of k in some position. The result is an identity in the determinants.

Take one of the R_1 and form a new $3n$ rowed determinant

$$\begin{array}{l|l|l}
1\ 2\ 3\ldots\ldots n & 0\ 0\ 0\ldots\ldots\ldots 1 & \alpha\ \beta\ \gamma\ldots\ldots\varsigma \\
1\ 2\ 3\ldots\ldots n & a\ b\ c\ldots\ldots y\ 1 & \alpha\ \beta\ \gamma\ldots\ldots\varsigma \\
1\ 2\ 3\ldots\ldots n & 0\ 0\ 0\ldots\ldots 0\ 0 & \alpha\ \beta\ \gamma\ldots\ldots\varsigma
\end{array}$$

Expanding by middle rows as before, we get $D\equiv\Sigma\, a\, R(k)+\Sigma\, b\, R'_n$. By lower rows, we reach

$$kR(\bar{k})\equiv\Sigma\, a\, R_2+\Sigma\, b\, R'_n+\Sigma\, c\, R(k).$$

Finally we reach R_{n-1}, and, by working on R'_{n-1} and on R'_n, we reduce them to R_{n-1}, *i. e.*, to relations involving $n-1$ rows of k not opposite the zero.

We then apply

$$\begin{array}{l|l|l}
1\ 2\ldots\ldots n & 0\ 0\ 0\ldots\ldots 0\ 1 & 2\ 3\ldots\ldots(n-1)\ \ \varepsilon\ \ \varsigma \\
1\ 2\ldots\ldots n & a\ b\ c\ldots\ldots y\ 1 & 2\ 3\ldots\ldots(n-1)\ \ \varepsilon\ \ \varsigma \\
1\ 2\ldots\ldots n & 0\ 0\ 0\ldots\ldots 0\ 0 & 0\ 0\ldots\ldots\ldots 0\ \ \ \ \ \ \ \varepsilon\ \ \varsigma
\end{array}$$

The $3n$ rowed determinant vanishes in the elements; for, subtraction of the upper n rows from middle leaves only $(n-1)$ columns. This time $D=\Sigma\, a\, R(k)+\Sigma\, b\, R_{n-1}$, when the symbols giving rise to R_{n-1} contain only two columns foreign to k and either vanishes identically or contains k. Expanding by lower rows, we get $D\equiv k\,R(\bar{k})-\Sigma\, a\, R(k)$. Hence each $R(\bar{k})$ is reduced by a chain to depend on $R(k)$.

Now assume that all the $R_3(k)$ and one $R_r(k)$ in each set of columns vanish. The number of relations assumed satisfied is I, the total number of independent relations. Apply a series of reducing operations to

$$R_r(k)+R'_r(k)=\Sigma\, R''_r(k).$$

The $R''_r(k)$ have already n columns of k, and one more is introduced not opposite the zero. The result is an extensional with one more column among the extenders, *i. e.*, is an $R''_{r-1}(\bar{k})$. Hence the $R'_r(k)$ do not reappear in the chain. In treating the $R(\bar{k})$ it may happen that an extender is replaced on one side and and $R_r(\bar{k})$ lead to $R_{r+1}(\bar{k})$ or even to an $R_{r+1}(k)$, but these are reducible again, and eventually reduce to $R_3(k)$.

We finally have the theorem: *If $k \neq 0$, every relation is satisfied if every $R_3(k)=0$, and one $R_r(k)=0$ for every r and every set of columns.*

The number of independent determinants ot order n in an $m \times n$ array is thus in general $D=n(m-n)+1$.

In the particular case of zero values, however, either all vanish or there is an $(n-1)\times n$ array in which not all the minors of order $(n-1)$ vanish. Hence Cayley's theorem holds: $D_0=m-n+1$. If the $m-n+1$ determinants with $n-1$ columns of this array in common vanish, all others vanish. A general proof follows. A determinant not containing columns 1, 2, ..., $n-1$ has at least two columns not in this set. Hence the relation

$$\frac{0\ 0 \ldots\ldots 0\ \alpha \mid \beta\ \gamma \ldots\ldots \varsigma\ M}{1\ 2 \ldots\ldots n-1\ \alpha \mid \beta\ \gamma \ldots\ldots \varsigma\ M}=0,$$

gives $(\alpha\,\beta\,\gamma \ldots\ldots \varsigma)(1\ 2\ 3\ (n-1)M)=\Sigma(1\ 2\ 3 \ldots\ldots n-1\ \alpha)(M\,\beta\,\gamma \ldots\ldots \varsigma)$ if M be an arbitrary column and every minor of order $n-1$ from $(1, 2, \ldots\ldots, n-1)$ be not zero, $(1, 2, \ldots\ldots, n-1, M)\neq 0$. Then from $(1, 2, 3, \ldots\ldots, n-1, \alpha)=0$, α any of $m-n+1$ columns, follows $(\alpha\ \beta\ \gamma \ldots\ldots \varsigma)=0$. As in the general case $m-n \geq 2$.

There are also relations between determinants of different orders taken from an array. The vanishing determinant

$$\begin{array}{r|ccccccc|}
 & a_1 & b_1 \ldots\ldots & f_1, & a_1 & b_1 & c_1 \ldots\ldots & l_1 \\
 & a_2 & b_2 \ldots\ldots & f_2, & a_2 & b_2 & c_1 \ldots\ldots & l_2 \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & a_{n+1} & b_{n+1} & f_{n+1}, & a_{n+1} & b_{n+1} & c_{n+1} \ldots\ldots & l_{n+1} \\
 & 0 & 0 \ldots\ldots & 0 & a_i & b_i & c_i \ldots\ldots & l_i \\
 r-1 \text{ rows}\left\{\begin{array}{c}\\ \\ \\ \end{array}\right. & 0 & 0 \ldots\ldots & 0 & a_j & b_j & c_j \ldots\ldots & l_j \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & 0 & 0 \ldots\ldots & 0 & a_k & b_k & c_k \ldots\ldots & l_k \\
 \multicolumn{1}{c}{} & \multicolumn{3}{c}{\underbrace{\hphantom{0 \qquad 0 \ldots\ldots 0}}_{r \text{ columns}}} & \multicolumn{4}{c}{}
\end{array}$$

gives r types of identities between $(n \times n)$ and $(r \times r)$ determinants. If s of the set $(i, j, \ldots\ldots, n+1)$, there are $\begin{bmatrix} n \\ r \end{bmatrix}\begin{bmatrix} m \\ n+1 \end{bmatrix}\begin{bmatrix} n+1 \\ s \end{bmatrix}\begin{bmatrix} m-n-1 \\ r-s-1 \end{bmatrix}$ of the type s $s=[0, 1, \ldots\ldots, (r-1)]$. If the array have M rows and N columns the total number of type s is $\begin{bmatrix} M \\ N \end{bmatrix}\begin{bmatrix} N \\ r \end{bmatrix}\begin{bmatrix} M \\ n+1 \end{bmatrix}\begin{bmatrix} n+1 \\ s \end{bmatrix}\begin{bmatrix} M-n-1 \\ r-s-1 \end{bmatrix}$. These relations between $(n\times n)$ and $(r\times r)$ determinants, $n>r$, can be reduced to sums of products of determinants of $(n-r)^2$ elements and the relations between determinants of the same order r added to products of $(n-r)$th order determinants and identities between determinants of order r which vanish in the determinants.

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SEVERAL FUNDAMENTAL THEOREMS IN GROUP THEORY.

By DR. G. A. MILLER.

The following theorems are readily deduced from known theorems. They appear to be of sufficient importance to be explicitly stated in view of their numerous elementary applications.

Theorem I. *If a group contains more than one cyclic subgroup of order* $k=p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots\ldots\ldots, p_\lambda^{\alpha_\lambda}$, *where* $p_1, p_2, \ldots\ldots\ldots, p_\lambda$ *are distinct prime numbers such that* $p_1<p_2<\ldots\ldots\ldots<p_\lambda$, *then it contains at least* p_1 *such subgroups. If the number of these subgroups is exactly* p_1 *the value of* α_1 *exceeds unity.*

This theorem may readily be derived from the known theorems that the number of cyclic subgroups of order p^α, $\alpha>1$ and $p>2$, in a non-cyclic group of order p^m is always divisible by p, and that the total number of subgroups of order p^α in such a group is of the form $1+mp$. Suppose that a group G contains more than one cyclic subgroup of order k but that the number of these subgroups is less than p_1+1. Each of these subgroups is transformed into itself by every other one, since an operator of order p^β transforms things in multiples of p when it does not transform them into themselves.

From this it follows that two of these cyclic subgroups generate a group which is the direct product of its Sylow subgroup. Since these Sylow subgroups cannot all be cyclic it is proved that this direct product contains at least p_1 cyclic subgroups of order k, and hence G contains at least p_1 cyclic subgroups of this order. From the fact that a group of order p^m cannot contain exactly p cyclic subgroups of order p^α when $\alpha=1$, it follows directly that $\alpha_1>1$ whenever G contains exactly p_1 cyclic subgroups of order k.

As an interesting particular consequence of this theorem we have that there is no group which contains exactly two cyclic subgroups of the same order when this order is either odd or twice an odd number. Moreover, it is easy to construct groups containing exactly two cyclic subgroups of any arbitrary order which is divisible by 4. For instance, there is an infinite number of groups containing just two cyclic subgroups of order twelve, but there is not a single